

# Pfaffian Expressions for Random Matrix Correlation Functions

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**Abstract** It is well known that Pfaffian formulas for eigenvalue correlations are useful in the analysis of real and quaternion random matrices. Moreover the parametric correlations in the crossover to complex random matrices are evaluated in the forms of Pfaffians. In this article, we review the formulations and applications of Pfaffian formulas. For that purpose, we first present the general Pfaffian expressions in terms of the corresponding skew orthogonal polynomials. Then we clarify the relation to Eynard and Mehta's determinant formula for hermitian matrix models and explain how the evaluation is simplified in the cases related to the classical orthogonal polynomials. Applications of Pfaffian formulas to random matrix theory and other fields are also mentioned.

**Keywords** Pfaffian · Random matrices · Orthogonal polynomials

## 1 Introduction

Pfaffian formulas were first introduced by Dyson to the theory of random matrices [1]. Extending Mehta and Gaudin's pioneering work [2], Dyson proved that the eigenvalue correlation functions for unitary random matrices were written in the forms of quaternion determinants, which were equivalent to Pfaffians. His method was then extended and applied to more general random matrix ensembles [3–8]. Among all, Pandey and Mehta's work [9–11] on the crossover between matrix symmetries revealed the wide applicability of Pfaffians. Extending Pandey and Mehta's result, Nagao and Forrester established more general Pfaffian formulas for parametric correlation functions [12]. Parametric correlation functions describe the correlations among the eigenvalues with different crossover parameters. Then Pfaffian formulas were furthermore extended to involve the multi-matrix models combining matrices of different symmetries [13]. In constructing Pfaffian formulas, the corresponding skew orthogonal polynomials play a crucial role. As a result, the skew orthogonal polynomials

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and related ordinary orthogonal polynomials are extensively used in the asymptotic analysis of the correlations.

Recently Pfaffian formulas found new applications in combinatorics and related problems in non-equilibrium statistical physics. Vicious random walkers [14–16], polynuclear growth models [17], sequences of partitions (the Pfaffian Schur process) [18] and random involutions [19] were explored and statistical fluctuations were analytically evaluated. These new applications are regarded as discretizations of random matrix ensembles and we often make use of the discrete versions of orthogonal polynomials.

In this article, we present the Pfaffian formulas employed in the analysis of eigenvalue correlations, list the important special cases and explain the relevance to random matrix theory and other applications. In Sect. 2, the general Pfaffian expressions are given in the forms of multiple integrals. In Sect. 3, we demonstrate that in a special case the Pfaffians are reduced to a simpler determinant expression. In Sect. 4, we explore simple cases in which the Pfaffian formulas become more tractable. In particular, we focus on the cases related to the (continuous and discrete) classical orthogonal polynomials and explicitly list the corresponding skew orthogonal polynomials. In Sect. 5, the relevance to random matrices and other new applications are briefly summarized.

To begin with, let us introduce the Pfaffian. For an antisymmetric  $N \times N$  matrix  $A$ , when  $N$  is even, the Pfaffian is defined as

$$\text{Pf}[A] = \frac{1}{(N/2)!} \sum'_P (-1)^P A_{j_1 j_2} A_{j_3 j_4} \cdots A_{j_{N-1} j_N}. \tag{1.1}$$

Here  $P = (j_1, j_2, \dots, j_N)$  denotes a permutation of  $(1, 2, \dots, N)$  and  $(-1)^P$  is the sign of  $P$ . The summation  $\sum'_P$  is taken over all  $P$  satisfying the restriction  $j_1 < j_2, j_3 < j_4, \dots, j_{N-1} < j_N$ .

In terms of the vectors

$$\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)} \tag{1.2}$$

with the elements

$$\mathbf{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \dots, x_N^{(j)}), \tag{1.3}$$

we define the functions

$$p(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)}) = \prod_{j>l}^N (x_j^{(M)} - x_l^{(M)}) \text{Pf}[F(x_j^{(1)}, x_l^{(1)})]_{j,l=1,2,\dots,N} \\ \times \prod_{m=1}^{M-1} \det[g^{(m)}(x_j^{(m+1)}, x_l^{(m)})]_{j,l=1,2,\dots,N} \quad \text{for even } N \tag{1.4}$$

and 
$$p(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)}) = \prod_{j>l}^N (x_j^{(M)} - x_l^{(M)}) \\ \times \text{Pf} \begin{bmatrix} [F(x_j^{(1)}, x_l^{(1)})]_{j,l=1,2,\dots,N} & [f(x_j^{(1)})]_{j=1,2,\dots,N} \\ -[f(x_l^{(1)})]_{l=1,2,\dots,N} & 0 \end{bmatrix} \\ \times \prod_{m=1}^{M-1} \det[g^{(m)}(x_j^{(m+1)}, x_l^{(m)})]_{j,l=1,2,\dots,N} \quad \text{for odd } N. \tag{1.5}$$

We suppose that the corresponding measure is given by

$$\prod_{j=1}^N d\mu_1(x_j^{(1)}) \prod_{j=1}^N d\mu_2(x_j^{(2)}) \cdots \prod_{j=1}^N d\mu_M(x_j^{(M)}). \tag{1.6}$$

The functions  $F(x, y)$ ,  $g^{(m)}(x, y)$  and  $f(x)$  are arbitrary besides the antisymmetry relation

$$F(x, y) = -F(y, x) \tag{1.7}$$

and the assumption that divergences are avoided in relevant calculations. As explained in Sect. 5, the functions  $p$  (with normalization constants multiplied) generalize the probability distribution functions of random matrix eigenvalues.

We are interested in the multiple integrals

$$\begin{aligned} &W(x_1^{(1)}, \dots, x_{k_1}^{(1)}; x_1^{(2)}, \dots, x_{k_2}^{(2)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) \\ &= \frac{1}{\prod_{m=1}^M (N - k_m)!} \int \prod_{j=k_1+1}^N d\mu_1(x_j^{(1)}) \int \prod_{j=k_2+1}^N d\mu_2(x_j^{(2)}) \cdots \int \prod_{j=k_M+1}^N d\mu_M(x_j^{(M)}) \\ &\quad \times p(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)}), \end{aligned} \tag{1.8}$$

which give the expressions for the eigenvalue correlations. The Pfaffian formulas for these multiple integrals are presented in next section.

## 2 Pfaffian Formulas for the Correlation Functions

### 2.1 The Case $N$ Even

Let us recursively define

$$G^{(m,n)}(x, y) = \begin{cases} \delta_m(x - y), & m = n, \\ \int d\mu_{m-1}(z) g^{(m-1)}(x, z) G^{(m-1,n)}(z, y), & m > n, \end{cases} \tag{2.1}$$

where  $\delta_m(x)$  is Dirac’s delta function with respect to the measure  $d\mu_m(x)$ . Then we introduce

$$F^{(m,n)}(x, y) = \int d\mu_1(x') \int d\mu_1(y') G^{(m,1)}(x, x') G^{(n,1)}(y, y') F(x', y') \tag{2.2}$$

and an antisymmetric inner product

$$\langle f, g \rangle^{(m)} = \int d\mu_m(x) \int d\mu_m(y) F^{(m,m)}(x, y) f(x) g(y). \tag{2.3}$$

By means of the antisymmetric inner product, we construct the monic skew orthogonal polynomials  $R_k^{(M)}(x) = x^k + \dots$ , which satisfy the skew orthogonality relation

$$\langle R_{2j}^{(M)}, R_{2l+1}^{(M)} \rangle^{(M)} = -\langle R_{2l+1}^{(M)}, R_{2j}^{(M)} \rangle^{(M)} = r_j \delta_{jl}, \tag{2.4}$$

$$\langle R_{2j}^{(M)}, R_{2l}^{(M)} \rangle^{(M)} = 0, \quad \langle R_{2j+1}^{(M)}, R_{2l+1}^{(M)} \rangle^{(M)} = 0. \tag{2.5}$$

Let us denote the monic monomial of order  $j$  by  $\Pi_j(x) = x^j$ . Then, with a notation

$$J_{jl} = \langle \Pi_j, \Pi_l \rangle^{(M)}, \tag{2.6}$$

we find the determinant expressions for  $R_k^{(M)}(x)$  as

$$R_{2k}^{(M)}(x) = \frac{1}{u_k} \begin{vmatrix} \Pi_{2k}(x) & J_{2k \ 2k-1} & J_{2k \ 2k-2} & \dots & J_{2k \ 0} \\ \Pi_{2k-1}(x) & J_{2k-1 \ 2k-1} & J_{2k-1 \ 2k-2} & \dots & J_{2k-1 \ 0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_0(x) & J_{0 \ 2k-1} & J_{0 \ 2k-2} & \dots & J_{0 \ 0} \end{vmatrix}, \tag{2.7}$$

$$R_{2k+1}^{(M)}(x) = \frac{1}{u_k} \begin{vmatrix} \Pi_{2k+1}(x) & J_{2k+1 \ 2k-1} & J_{2k+1 \ 2k-2} & \dots & J_{2k+1 \ 0} \\ \Pi_{2k-1}(x) & J_{2k-1 \ 2k-1} & J_{2k-1 \ 2k-2} & \dots & J_{2k-1 \ 0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_0(x) & J_{0 \ 2k-1} & J_{0 \ 2k-2} & \dots & J_{0 \ 0} \end{vmatrix} + v_k R_{2k}^{(M)}(x), \tag{2.8}$$

where

$$u_k = \begin{vmatrix} J_{2k-1 \ 2k-1} & J_{2k-1 \ 2k-2} & \dots & J_{2k-1 \ 0} \\ J_{2k-2 \ 2k-1} & J_{2k-2 \ 2k-2} & \dots & J_{2k-2 \ 0} \\ \vdots & \vdots & \ddots & \vdots \\ J_{0 \ 2k-1} & J_{0 \ 2k-2} & \dots & J_{0 \ 0} \end{vmatrix} \tag{2.9}$$

and  $v_k$  is an arbitrary constant. Note the indeterminacy of  $R_{2k+1}^{(M)}(x)$  due to  $v_k$ .

Moreover we introduce

$$R_k^{(m)}(x) = \int d\mu_M(y) R_k^{(M)}(y) G^{(M,m)}(y, x), \tag{2.10}$$

$$\Phi_k^{(m)}(x) = \int d\mu_m(y) R_k^{(m)}(y) F^{(m,m)}(y, x) \tag{2.11}$$

and define the matrices  $D^{(m,n)}$ ,  $I^{(m,n)}$ ,  $S^{(m,n)}$  as

$$D_{j,l}^{(m,n)} = \sum_{k=0}^{(N/2)-1} \frac{1}{r_k} \left[ R_{2k}^{(m)}(x_j^{(m)}) R_{2k+1}^{(n)}(x_l^{(n)}) - R_{2k+1}^{(m)}(x_j^{(m)}) R_{2k}^{(n)}(x_l^{(n)}) \right], \tag{2.12}$$

$$I_{j,l}^{(m,n)} = - \sum_{k=0}^{(N/2)-1} \frac{1}{r_k} \left[ \Phi_{2k}^{(m)}(x_j^{(m)}) \Phi_{2k+1}^{(n)}(x_l^{(n)}) - \Phi_{2k+1}^{(m)}(x_j^{(m)}) \Phi_{2k}^{(n)}(x_l^{(n)}) \right] + F_{j,l}^{(m,n)}, \tag{2.13}$$

$$S_{j,l}^{(m,n)} = \sum_{k=0}^{(N/2)-1} \frac{1}{r_k} \left[ \Phi_{2k}^{(m)}(x_j^{(m)}) R_{2k+1}^{(n)}(x_l^{(n)}) - \Phi_{2k+1}^{(m)}(x_j^{(m)}) R_{2k}^{(n)}(x_l^{(n)}) \right] - G_{j,l}^{(m,n)}, \tag{2.14}$$

where

$$F_{j,l}^{(m,n)} = F^{(m,n)}(x_j^{(m)}, x_l^{(n)}) \tag{2.15}$$

and

$$G_{j,l}^{(m,n)} = \begin{cases} 0, & m \leq n, \\ G^{(m,n)}(x_j^{(m)}, x_l^{(n)}), & m > n. \end{cases} \tag{2.16}$$

Let us consider an antisymmetric matrix  $A$  which consists of matrix blocks  $A^{(m,n)}$ ,  $m, n = 1, 2, \dots, M$ . Each block  $A^{(m,n)}$  is a  $2N \times 2N$  matrix which consists of  $2 \times 2$  blocks

$$A_{j,l}^{(m,n)} = \begin{pmatrix} D_{j,l}^{(m,n)} & S_{l,j}^{(n,m)} \\ -S_{j,l}^{(m,n)} & -I_{j,l}^{(m,n)} \end{pmatrix}, \quad j, l = 1, 2, \dots, N. \tag{2.17}$$

Then it turns out that the Pfaffian expression for the multiple integral (1.8) is [12, 13]

$$\begin{aligned} W(x_1^{(1)}, \dots, x_{k_1}^{(1)}; x_1^{(2)}, \dots, x_{k_2}^{(2)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) \\ = \prod_{j=1}^{N/2} r_{j-1} \text{Pf}[A^{(m,n)}(k_m, k_n)]_{m,n=1,2,\dots,M}. \end{aligned} \tag{2.18}$$

Here each block  $A^{(m,n)}(k_m, k_n)$  is obtained from  $A^{(m,n)}$  by removing the  $2k_m + 1, 2k_m + 2, \dots, 2N$ th rows and  $2k_n + 1, 2k_n + 2, \dots, 2N$ th columns. The proof of this Pfaffian expression is quite similar to that in [13], although it is given here in a slightly generalized form.

### 2.2 The Case $N$ Odd

In terms of the functions  $G^{(m,n)}(x, y)$  and  $R_k^{(m)}(x)$  defined in previous subsection, let us employ the notations

$$f^{(m)}(x) = \int d\mu_1(x') G^{(m,1)}(x, x') f(x') \tag{2.19}$$

and

$$s_k = \int d\mu_M(y) R_k^{(M)}(y) f^{(M)}(y). \tag{2.20}$$

Moreover we introduce

$$\bar{R}_k^{(m)}(x) = R_k^{(m)}(x) - \frac{S_k}{S_{N-1}} R_{N-1}^{(m)}(x), \quad k = 0, 1, 2, \dots, N - 2 \tag{2.21}$$

and

$$\begin{aligned} \bar{\Phi}_k^{(m)}(x) &= \int d\mu_m(y) \bar{R}_k^{(m)}(y) F^{(m,m)}(y, x) \\ &= \Phi_k^{(m)}(x) - \frac{S_k}{S_{N-1}} \Phi_{N-1}^{(m)}(x), \quad k = 0, 1, 2, \dots, N - 2. \end{aligned} \tag{2.22}$$

Then the matrices  $\bar{D}^{(m,n)}$ ,  $\bar{I}^{(m,n)}$ ,  $\bar{S}^{(m,n)}$  are defined as

$$\bar{D}_{j,l}^{(m,n)} = \sum_{k=0}^{(N-3)/2} \frac{1}{r_k} \left[ \bar{R}_{2k}^{(m)}(x_j^{(m)}) \bar{R}_{2k+1}^{(n)}(x_l^{(n)}) - \bar{R}_{2k+1}^{(m)}(x_j^{(m)}) \bar{R}_{2k}^{(n)}(x_l^{(n)}) \right], \tag{2.23}$$

$$\begin{aligned} \bar{I}_{j,l}^{(m,n)} &= - \sum_{k=0}^{(N-3)/2} \frac{1}{r_k} \left[ \bar{\Phi}_{2k}^{(m)}(x_j^{(m)}) \bar{\Phi}_{2k+1}^{(n)}(x_l^{(n)}) - \bar{\Phi}_{2k+1}^{(m)}(x_j^{(m)}) \bar{\Phi}_{2k}^{(n)}(x_l^{(n)}) \right] \\ &+ \frac{1}{S_{N-1}} \left[ \Phi_{N-1}^{(m)}(x_j^{(m)}) f^{(n)}(x_l^{(n)}) - \Phi_{N-1}^{(n)}(x_l^{(n)}) f^{(m)}(x_j^{(m)}) \right] + F_{j,l}^{(m,n)}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} \bar{S}_{j,l}^{(m,n)} &= \sum_{k=0}^{(N-3)/2} \frac{1}{r_k} \left[ \bar{\Phi}_{2k}^{(m)}(x_j^{(m)}) \bar{R}_{2k+1}^{(n)}(x_l^{(n)}) - \bar{\Phi}_{2k+1}^{(m)}(x_j^{(m)}) \bar{R}_{2k}^{(n)}(x_l^{(n)}) \right] \\ &+ \frac{1}{s_{N-1}} f^{(m)}(x_j^{(m)}) R_{N-1}^{(n)}(x_l^{(n)}) - G_{j,l}^{(m,n)}. \end{aligned} \tag{2.25}$$

As before we construct an antisymmetric matrix  $\bar{A}$  so that it consists of matrix blocks  $\bar{A}^{(m,n)}$ ,  $m, n = 1, 2, \dots, M$  and each block consists of  $2 \times 2$  blocks

$$\bar{A}_{j,l}^{(m,n)} = \begin{pmatrix} \bar{D}_{j,l}^{(m,n)} & \bar{S}_{l,j}^{(n,m)} \\ -\bar{S}_{j,l}^{(m,n)} & -\bar{I}_{j,l}^{(m,n)} \end{pmatrix}, \quad j, l = 1, 2, \dots, N. \tag{2.26}$$

It follows that the Pfaffian expression for the multiple integral (1.8) is written as [12, 13]

$$\begin{aligned} W(x_1^{(1)}, \dots, x_{k_1}^{(1)}; x_1^{(2)}, \dots, x_{k_2}^{(2)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) \\ = s_{N-1} \prod_{j=1}^{(N-1)/2} r_{j-1} \text{Pf}[\bar{A}^{(m,n)}(k_m, k_n)]_{m,n=1,2,\dots,M}. \end{aligned} \tag{2.27}$$

One obtains each block  $\bar{A}^{(m,n)}(k_m, k_n)$  by removing the  $2k_m + 1, 2k_m + 2, \dots, 2N$ th rows and  $2k_n + 1, 2k_n + 2, \dots, 2N$ th columns from the block  $\bar{A}^{(m,n)}$ . It is again straightforward to prove this Pfaffian formula by following the procedures in [13].

We remark that the above Pfaffian formulas can be extended to the case with the factor

$$\prod_{j>l}^N (x_j^{(M)} - x_l^{(M)}) = \det[(x_j^{(M)})^{l-1}]_{j,l=1,2,\dots,N} \tag{2.28}$$

(in (1.4) and (1.5)) replaced by a general determinant factor [18]

$$\det[\varphi_{l-1}(x_j^{(M)})]_{j,l=1,2,\dots,N}. \tag{2.29}$$

In that case we need to replace  $\Pi_j(x)$  by  $\varphi_j(x)$  in (2.6–2.8). The resulting skew orthogonal functions  $R_k^{(M)}(x)$  are not in general polynomials.

### 3 Reduction to Eynard–Mehta Formula

In this section we consider the special case

$$g^{(1)}(x, y) = \sum_{j=0}^{N-1} \frac{1}{h_j} Q_j(x) R_j(y), \tag{3.1}$$

where  $Q_j(x) = x^j + \dots$  and  $R_j(x) = x^j + \dots$  are monic polynomials. The determinant of this function can be rewritten as

$$\begin{aligned} \det[g^{(1)}(x_j, y_l)]_{j,l=1,2,\dots,N} \\ = \frac{1}{\prod_{j=0}^{N-1} h_j} \det[Q_{j-1}(x_l)]_{j,l=1,2,\dots,N} \det[R_{j-1}(y_l)]_{j,l=1,2,\dots,N} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\prod_{j=0}^{N-1} h_j} \det[(x_l)^{j-1}]_{j,l=1,2,\dots,N} \det[(y_l)^{j-1}]_{j,l=1,2,\dots,N} \\
 &= \frac{1}{\prod_{j=0}^{N-1} h_j} \prod_{j<l}^N (x_j - x_l) \prod_{j<l}^N (y_j - y_l),
 \end{aligned}
 \tag{3.2}$$

so that both of the functions (1.4) and (1.5) are decomposed into two factors as

$$p(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)}) = p^{(I)}(\mathbf{x}^{(1)}) p^{(II)}(\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)}).
 \tag{3.3}$$

The second factor

$$\begin{aligned}
 p^{(II)}(\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)}) &= \frac{1}{\prod_{j=0}^{N-1} h_j} \prod_{j>l}^N (x_j^{(M)} - x_l^{(M)})(x_j^{(2)} - x_l^{(2)}) \\
 &\quad \times \prod_{m=2}^{M-1} \det[g^{(m)}(x_j^{(m+1)}, x_l^{(m)})]_{j,l=1,2,\dots,N}
 \end{aligned}
 \tag{3.4}$$

has the form of the probability distribution function for the hermitian multi-matrix models. Eynard and Mehta derived a determinant formula for the eigenvalue correlations of such hermitian multi-matrix models [20]. Therefore, in this special case, their determinant formula should be reclaimed from the Pfaffian formulas.

In order to see the reduction to Eynard and Mehta’s formula, we choose  $Q_j(x)$  and constants  $h_j$  so that the orthogonality relation

$$\int d\mu_M(x) d\mu_2(y) G^{(M,2)}(x, y) P_j(x) Q_l(y) = h_j \delta_{jl}
 \tag{3.5}$$

holds. Here  $P_j(x) = x^j + \dots$  are monic polynomials. Moreover we define

$$P_j^{(m)}(x) = \int d\mu_M(y) P_j(y) G^{(M,m)}(y, x), \quad 1 \leq m \leq M
 \tag{3.6}$$

and

$$Q_j^{(m)}(x) = \int d\mu_2(y) G^{(m,2)}(x, y) Q_j(y), \quad 2 \leq m \leq M.
 \tag{3.7}$$

Then it can readily be seen that

$$\int d\mu_m(x) \int d\mu_n(y) G^{(m,n)}(x, y) P_j^{(m)}(x) Q_l^{(n)}(y) = h_j \delta_{jl}.
 \tag{3.8}$$

Let us use (3.1) to find

$$\begin{aligned}
 G^{(m,1)}(x, y) &= \int d\mu_2(z) G^{(m,2)}(x, z) g^{(1)}(z, y) \\
 &= \sum_{j=0}^{N-1} \frac{1}{h_j} Q_j^{(m)}(x) R_j(y), \quad 2 \leq m \leq M.
 \end{aligned}
 \tag{3.9}$$

Then we obtain

$$\begin{aligned}
 F^{(m,n)}(x, y) &= \int d\mu_1(x') \int d\mu_1(y') G^{(m,1)}(x, x') G^{(n,1)}(y, y') F(x', y') \\
 &= \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \frac{1}{h_j h_l} Q_j^{(m)}(x) Q_l^{(n)}(y) \langle R_j, R_l \rangle^{(1)}, \quad 2 \leq m, n \leq M, \quad (3.10)
 \end{aligned}$$

from which it follows that

$$\int d\mu_M(x) \int d\mu_M(y) F^{(M,M)}(x, y) P_j^{(M)}(x) P_l^{(M)}(y) = \langle R_j, R_l \rangle^{(1)}. \quad (3.11)$$

Let us choose  $R_k(x)$  so that the skew orthogonality relation

$$\langle R_{2j}, R_{2l+1} \rangle^{(1)} = -\langle R_{2l+1}, R_{2j} \rangle^{(1)} = r_j \delta_{jl}, \quad (3.12)$$

$$\langle R_{2j}, R_{2l} \rangle^{(1)} = 0, \quad \langle R_{2j+1}, R_{2l+1} \rangle^{(1)} = 0 \quad (3.13)$$

holds. Then it can immediately be seen from (3.11) that

$$R_j^{(M)}(x) = P_j^{(M)}(x), \quad (3.14)$$

which yields

$$R_j^{(m)}(x) = P_j^{(m)}(x), \quad m = 1, 2, \dots, M \quad (3.15)$$

in general. Moreover we find

$$\begin{aligned}
 \Phi_j^{(m)}(x) &= \int d\mu_m(y) P_j^{(m)}(y) F^{(m,m)}(y, x) \\
 &= \sum_{l=0}^{N-1} \frac{1}{h_l} Q_l^{(m)}(x) \langle R_j, R_l \rangle^{(1)}, \quad 2 \leq m \leq M, \quad (3.16)
 \end{aligned}$$

which means

$$\Phi_{2j}^{(m)}(x) = \frac{r_j}{h_{2j+1}} Q_{2j+1}^{(m)}(x), \quad 2j \leq N - 2, \quad 2 \leq m \leq M, \quad (3.17)$$

$$\Phi_{2j+1}^{(m)}(x) = -\frac{r_j}{h_{2j}} Q_{2j}^{(m)}(x), \quad 2j + 1 \leq N, \quad 2 \leq m \leq M.$$

Therefore, for even  $N$ , it follows from (2.13) and (3.17) that

$$I_{j,l}^{(m,n)} = 0, \quad 2 \leq m, n \leq M \quad (3.18)$$

and

$$I_{j,l}^{(m,1)} = -I_{l,j}^{(1,m)} = 0, \quad 2 \leq m \leq M. \quad (3.19)$$

Moreover, from (2.14) and (3.17), we obtain

$$S_{j,l}^{(m,n)} = \sum_{k=0}^{N-1} \frac{1}{h_k} Q_k^{(m)}(x_j^{(m)}) P_k^{(n)}(x_l^{(n)}) - G_{j,l}^{(m,n)}, \quad 2 \leq m, n \leq M \quad (3.20)$$



and

$$S_{j,l}^{(m,1)} = 0, \quad m \geq 2. \tag{3.21}$$

We are now in a position to examine the Pfaffian expression (2.18). It follows from the properties of the Pfaffian that the elements of the blocks  $A^{(m,n)}(k_m, k_n)$

$$A_{j,l}^{(m,n)} = \begin{cases} \begin{pmatrix} D_{j,l}^{(m,n)} & S_{l,j}^{(n,m)} \\ -S_{j,l}^{(m,n)} & 0 \end{pmatrix}, & m, n \geq 2, \\ \begin{pmatrix} D_{j,l}^{(m,n)} & S_{l,j}^{(n,m)} \\ 0 & 0 \end{pmatrix}, & m \geq 2, n = 1, \\ \begin{pmatrix} D_{j,l}^{(m,n)} & 0 \\ -S_{j,l}^{(m,n)} & 0 \end{pmatrix}, & m = 1, n \geq 2 \end{cases} \tag{3.22}$$

can be replaced by

$$A_{j,l}^{(m,n)} = \begin{cases} \begin{pmatrix} 0 & S_{l,j}^{(n,m)} \\ -S_{j,l}^{(m,n)} & 0 \end{pmatrix}, & m, n \geq 2, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & m \geq 2, n = 1, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & m = 1, n \geq 2 \end{cases} \tag{3.23}$$

without changing the value of the Pfaffian. Therefore it is straightforward to see that

$$\begin{aligned} & \text{Pf}[A^{(m,n)}(k_m, k_n)]_{m,n=1,2,\dots,M} \\ &= \text{Pf}[A^{(1,1)}(k_1, k_1)] \text{Pf} \left[ \begin{array}{cc} 0 & S_{l,j}^{(n,m)} \\ -S_{j,l}^{(m,n)} & 0 \end{array} \right]_{j=1,\dots,k_m, l=1,\dots,k_n, m=2,\dots,M, n=2,\dots,M} \\ &= \text{Pf}[A^{(1,1)}(k_1, k_1)] \det[S_{j,l}^{(m,n)}]_{j=1,\dots,k_m, l=1,\dots,k_n, m=2,\dots,M, n=2,\dots,M}. \end{aligned} \tag{3.24}$$

The expression (3.20) of  $S_{j,l}^{(m,n)}$  reveals that the second determinant factor in the last line of the above equation gives Eynard and Mehta’s formula.

For odd  $N$ , (3.16) yields

$$\Phi_{N-1}^{(m)}(x) = 0, \quad 2 \leq m \leq M, \tag{3.25}$$

so that

$$\bar{\Phi}_k^{(m)}(x) = \Phi_k^{(m)}(x), \quad 2 \leq m \leq M, \tag{3.26}$$

from which we can readily find

$$\bar{I}_{j,l}^{(m,n)} = 0, \quad 2 \leq m, n \leq M, \tag{3.27}$$

$$\bar{I}_{j,l}^{(m,1)} = -\bar{I}_{l,j}^{(1,m)} = 0, \quad 2 \leq m \leq M, \tag{3.28}$$

$$\bar{S}_{j,l}^{(m,n)} = \sum_{k=0}^{N-1} \frac{1}{h_k} Q_k^{(m)}(x_j^{(m)}) P_k^{(n)}(x_l^{(n)}) - G_{j,l}^{(m,n)}, \quad 2 \leq m, n \leq M \tag{3.29}$$

and

$$\bar{S}_{j,l}^{(m,1)} = 0, \quad m \geq 2. \tag{3.30}$$

Following the steps as before, we can readily see that

$$\begin{aligned} & \text{Pf}[\bar{A}^{(m,n)}(k_m, k_n)]_{m,n=1,2,\dots,M} \\ &= \text{Pf}[\bar{A}^{(1,1)}(k_1, k_1)] \det[\bar{S}_{j,l}^{(m,n)}]_{j=1,\dots,k_m,l=1,\dots,k_n,m=2,\dots,M,n=2,\dots,M}, \end{aligned} \tag{3.31}$$

which gives Eynard and Mehta’s formula.

Thus we have shown that Eynard and Mehta’s determinant formula in [20] can be derived from the special case (3.1) of the Pfaffian formulas in Sect. 2.

### 4 Simple Special Cases

Let us go back to the general Pfaffian formulas in Sect. 2. The key ingredients of the Pfaffian expressions are the functions  $R_j^{(m)}(x)$  generated by (2.10) from the monic skew orthogonal polynomials  $R_j^{(M)}(x)$ . We introduce the monic orthogonal polynomials  $C_j^{(m)}(x) = x^j + \dots$  with the orthogonality relation

$$\int d\mu_m(x) C_j^{(m)}(x) C_l^{(m)}(x) = h_j^{(m)} \delta_{jl}. \tag{4.1}$$

Now, if the function  $g^{(m)}(x, y)$  is expanded as

$$g^{(m)}(x, y) = \sum_{j=0}^{\infty} \frac{\gamma_j^{(m+1)}}{\gamma_j^{(m)}} \frac{C_j^{(m+1)}(x) C_j^{(m)}(y)}{\sqrt{h_j^{(m+1)} h_j^{(m)}}}, \tag{4.2}$$

the evaluations of the integrals in (2.1) and (2.10) are greatly simplified [12, 21]. In this section, we assume this simplifying property.

By means of the expansion (4.2),  $G^{(m,n)}(x, y)$  is evaluated as

$$G^{(m,n)}(x, y) = \sum_{j=0}^{\infty} \frac{\gamma_j^{(m)}}{\gamma_j^{(n)}} \frac{C_j^{(m)}(x) C_j^{(n)}(y)}{\sqrt{h_j^{(m)} h_j^{(n)}}}, \tag{4.3}$$

where we assume the completeness of  $C_j^{(m)}(x)$

$$\delta_m(x - y) = \sum_{j=0}^{\infty} \frac{C_j^{(m)}(x) C_j^{(m)}(y)}{h_j^{(m)}}. \tag{4.4}$$

Therefore, if we expand  $R_k^{(M)}(x)$  in terms of  $C_j^{(M)}(x)$  as

$$R_k^{(M)}(x) = \sum_{j=0}^k \alpha_{kj} \frac{C_j^{(M)}(x)}{\gamma_j^{(M)} \sqrt{h_j^{(M)}}}, \quad \alpha_{kk} = \gamma_k^{(M)} \sqrt{h_k^{(M)}}, \tag{4.5}$$

it can be readily seen that

$$R_k^{(m)}(x) = \sum_{j=0}^k \alpha_{kj} \frac{C_j^{(m)}(x)}{\gamma_j^{(m)} \sqrt{h_j^{(m)}}}, \quad m = 1, 2, \dots, M \tag{4.6}$$

in general. Note that  $\alpha_{kj}$  are independent of  $m$ . Therefore, if  $\alpha_{kj}$  are known,  $R_k^{(m)}(x)$  are specified for all  $m$ . It is also possible to write down the inverse expansion as

$$\frac{C_k^{(m)}(x)}{\gamma_k^{(m)} \sqrt{h_k^{(m)}}} = \sum_{j=0}^k \beta_{kj} R_j^{(m)}(x). \tag{4.7}$$

Putting (4.3) into (2.2) leads to

$$F^{(m,n)}(x, y) = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{\gamma_j^{(m)} \gamma_l^{(n)}}{\gamma_j^{(1)} \gamma_l^{(1)}} \frac{C_j^{(m)}(x) C_l^{(n)}(y)}{\sqrt{h_j^{(m)} h_j^{(1)} h_l^{(n)} h_l^{(1)}}} \langle C_j^{(1)}, C_l^{(1)} \rangle^{(1)}. \tag{4.8}$$

Substituting (4.8) into (2.11) and using the expansions (4.6) and (4.7), we obtain

$$\begin{aligned} \Phi_k^{(m)}(x) &= \sum_{l=0}^{\infty} \frac{\gamma_l^{(m)}}{\gamma_l^{(1)}} \frac{C_l^{(m)}(x)}{\sqrt{h_l^{(m)} h_l^{(1)}}} \langle R_k^{(1)}, C_l^{(1)} \rangle^{(1)} \\ &= \sum_{v=0}^{\infty} \sum_{l=v}^{\infty} \gamma_l^{(m)} \frac{C_l^{(m)}(x)}{\sqrt{h_l^{(m)}}} \beta_{lv} \langle R_k^{(1)}, R_v^{(1)} \rangle^{(1)}, \end{aligned} \tag{4.9}$$

from which it follows that

$$\begin{aligned} \Phi_{2k}^{(m)}(x) &= \sum_{l=2k+1}^{\infty} \gamma_l^{(m)} \frac{C_l^{(m)}(x)}{\sqrt{h_l^{(m)}}} \beta_{l, 2k+1} r_k, \\ \Phi_{2k+1}^{(m)}(x) &= - \sum_{l=2k}^{\infty} \gamma_l^{(m)} \frac{C_l^{(m)}(x)}{\sqrt{h_l^{(m)}}} \beta_{l, 2k} r_k. \end{aligned} \tag{4.10}$$

For even  $N$ , let us put (4.10) into (2.14). After some algebra we arrive at

$$\begin{aligned} S_{j,l}^{(m,n)} &= \sum_{k=0}^{N-1} \frac{\gamma_k^{(m)} C_k^{(m)}(x_j^{(m)}) C_k^{(n)}(x_l^{(n)})}{\gamma_k^{(n)} \sqrt{h_k^{(m)} h_k^{(n)}}} \\ &+ \sum_{k=0}^{N-1} \sum_{v=N}^{\infty} \gamma_v^{(m)} \frac{C_v^{(m)}(x_j^{(m)})}{\sqrt{h_v^{(m)}}} \beta_{vk} R_k^{(n)}(x_l^{(n)}) - G_{j,l}^{(m,n)}. \end{aligned} \tag{4.11}$$

We also see that the expansion (4.7) can be substituted into (4.8). A comparison with (4.10) yields

$$F^{(m,n)}(x, y) = \sum_{k=0}^{\infty} \frac{1}{r_k} \left[ \Phi_{2k}^{(m)}(x) \Phi_{2k+1}^{(n)}(y) - \Phi_{2k+1}^{(m)}(x) \Phi_{2k}^{(n)}(y) \right], \tag{4.12}$$

so that

$$I_{j,l}^{(m,n)} = \sum_{k=N/2}^{\infty} \frac{1}{r_k} \left[ \Phi_{2k}^{(m)}(x_j^{(m)}) \Phi_{2k+1}^{(n)}(x_l^{(n)}) - \Phi_{2k+1}^{(m)}(x_j^{(m)}) \Phi_{2k}^{(n)}(x_l^{(n)}) \right]. \tag{4.13}$$

For odd  $N$ , noting

$$f^{(m)}(x) = \sum_{k=0}^{\infty} s_k \sum_{v=k}^{\infty} \gamma_v^{(m)} \frac{C_v^{(m)}(x)}{\sqrt{h_v^{(m)}}} \beta_{vk} = \sum_{k=0}^{\infty} \frac{1}{r_k} \left[ \Phi_{2k}^{(m)}(x) s_{2k+1} - \Phi_{2k+1}^{(m)}(x) s_{2k} \right], \tag{4.14}$$

we can readily find

$$\begin{aligned} \bar{S}_{j,l}^{(m,n)} &= \sum_{k=0}^{N-1} \frac{\gamma_k^{(m)} C_k^{(m)}(x_j^{(m)}) C_k^{(n)}(x_l^{(n)})}{\gamma_k^{(n)} \sqrt{h_k^{(m)} h_k^{(n)}}} \\ &+ \sum_{k=0}^{N-1} \sum_{v=N}^{\infty} \gamma_v^{(m)} \frac{C_v^{(m)}(x_j^{(m)})}{\sqrt{h_v^{(m)}}} \beta_{vk} R_k^{(n)}(x_l^{(n)}) - G_{j,l}^{(m,n)} \\ &- \frac{\Phi_{N-1}^{(m)}(x)}{s_{N-1}} \sum_{k=0}^{(N-3)/2} \frac{1}{r_k} \left[ s_{2k} R_{2k+1}^{(n)}(y) - s_{2k+1} R_{2k}^{(n)}(y) \right] \\ &+ \frac{R_{N-1}^{(n)}(y)}{s_{N-1}} \sum_{k=N}^{\infty} s_k \sum_{v=k}^{\infty} \gamma_v^{(m)} \frac{C_v^{(m)}(x)}{\sqrt{h_v^{(m)}}} \beta_{vk} \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \bar{I}_{j,l}^{(m,n)} &= \sum_{k=(N+1)/2}^{\infty} \frac{1}{r_k} \left[ \Phi_{2k}^{(m)}(x_j^{(m)}) \Phi_{2k+1}^{(n)}(x_l^{(n)}) - \Phi_{2k+1}^{(m)}(x_j^{(m)}) \Phi_{2k}^{(n)}(x_l^{(n)}) \right] \\ &- \frac{\Phi_{N-1}^{(m)}(x)}{s_{N-1}} \sum_{k=(N+1)/2}^{\infty} \frac{1}{r_k} \left[ s_{2k} \Phi_{2k+1}^{(n)}(x_l^{(n)}) - s_{2k+1} \Phi_{2k}^{(n)}(x_l^{(n)}) \right] \\ &- \frac{\Phi_{N-1}^{(n)}(y)}{s_{N-1}} \sum_{k=(N+1)/2}^{\infty} \frac{1}{r_k} \left[ \Phi_{2k}^{(m)}(x_j^{(m)}) s_{2k+1} - \Phi_{2k+1}^{(m)}(x_j^{(m)}) s_{2k} \right]. \end{aligned} \tag{4.16}$$

Thus we have shown that the Pfaffian formulas are expressed in terms of the expansion coefficients  $\alpha_{jl}$  and  $\beta_{jl}$ . Let us explain how  $\alpha_{jl}$  can be calculated. Suppose that  $\tilde{R}_k(x) = x^k + \dots$  are the monic skew orthogonal polynomials with the skew orthogonality relation

$$\langle \tilde{R}_{2j}, \tilde{R}_{2l+1} \rangle^{(1)} = -\langle \tilde{R}_{2l+1}, \tilde{R}_{2j} \rangle^{(1)} = \tilde{r}_j \delta_{jl}, \tag{4.17}$$

$$\langle \tilde{R}_{2j}, \tilde{R}_{2l} \rangle^{(1)} = 0, \quad \langle \tilde{R}_{2j+1}, \tilde{R}_{2l+1} \rangle^{(1)} = 0 \tag{4.18}$$

and that  $\tilde{R}_k(x)$  are expanded in terms of  $C_j^{(1)}(x)$  as

$$\tilde{R}_k(x) = \sum_{j=0}^k \tilde{\alpha}_{kj} C_j^{(1)}(x), \quad \tilde{\alpha}_{kk} = 1. \tag{4.19}$$

Since  $R_k^{(1)}(x)$  should equal to  $\tilde{R}_k(x)$  multiplied by a constant  $c_k$ , we can see that

$$\alpha_{kj} = \tilde{\alpha}_{kj} \gamma_j^{(1)} \sqrt{h_j^{(1)}} c_k, \quad r_k = c_{2k} c_{2k+1} \tilde{r}_k. \tag{4.20}$$

Noting  $\alpha_{kk} = \gamma_k^{(M)} \sqrt{h_k^{(M)}}$ , we find

$$c_k = \frac{\gamma_k^{(M)} \sqrt{h_k^{(M)}}}{\gamma_k^{(1)} \sqrt{h_k^{(1)}}}. \tag{4.21}$$

Thus the coefficients  $\alpha_{kj}$  can be calculated from the expansion (4.19).

Let us list the cases related to the classical orthogonal polynomials in which the expansions (4.19) are explicitly known. The inverse expansions are sometimes shown instead of (4.19). The inversions of the expansions are easy in all of the listed cases. For simplicity an abbreviated notations  $C_n(x)$  and  $h_n$  are used for  $C_n^{(1)}(x)$  and  $h_n^{(1)}$ , respectively. The sign function  $\text{sgn}(x)$  is defined as

$$\text{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases} \tag{4.22}$$

(1) *Hermite* [6–8] When the measure is given by

$$\int d\mu_1(x) = \int_{-\infty}^{\infty} dx e^{-x^2}, \tag{4.23}$$

the corresponding monic orthogonal polynomials are

$$C_n(x) = \frac{1}{2^n} H_n(x), \quad h_n = \frac{\sqrt{\pi} n!}{2^n}. \tag{4.24}$$

Here  $H_n(x)$  are the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{4.25}$$

The monic skew orthogonal polynomials  $\tilde{R}_n(x)$  are given as follows.

Case I:

$$\begin{aligned} F(x, y) &= e^{(x^2+y^2)/2} \text{sgn}(y-x), \\ \tilde{R}_{2n}(x) &= C_{2n}(x), \quad \tilde{R}_{2n+1}(x) = C_{2n+1}(x) - n C_{2n-1}(x), \\ \tilde{r}_n &= 2^{-2n+1} \sqrt{\pi} (2n)!. \end{aligned} \tag{4.26}$$

Case II:

$$\begin{aligned} F(x, y) &= 2e^{(x^2+y^2)/2} \frac{\partial}{\partial x} \delta(x-y), \\ C_{2n}(x) &= \tilde{R}_{2n}(x) - n \tilde{R}_{2n-2}(x), \quad C_{2n+1}(x) = \tilde{R}_{2n+1}, \\ \tilde{r}_n &= 2^{-2n} \sqrt{\pi} (2n+1)!. \end{aligned} \tag{4.27}$$

(2) *Laguerre* [6, 8, 22, 23] The measure is

$$\int d\mu_1(x) = \int_0^\infty dx x^a e^{-x}. \tag{4.28}$$

The corresponding monic orthogonal polynomials are

$$C_n(x) = (-1)^n n! L_n^{(a)}(x), \quad h_n = n!(n + a)!, \tag{4.29}$$

where  $L_n^{(a)}(x)$  are the Laguerre polynomials

$$L_n^{(a)}(x) = \frac{x^{-a} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+a}). \tag{4.30}$$

Case I:

$$\begin{aligned} F(x, y) &= x^{-(a+1)/2} y^{-(a+1)/2} e^{(x+y)/2} \operatorname{sgn}(y - x), \\ \tilde{R}_{2n}(x) &= C_{2n}(x), \\ \tilde{R}_{2n+1}(x) &= C_{2n+1}(x) - (2n)(a + 2n)C_{2n-1}(x), \\ \tilde{r}_n &= 4(2n)!(a + 2n)!. \end{aligned} \tag{4.31}$$

Case II:

$$\begin{aligned} F(x, y) &= 2x^{-(a-1)/2} y^{-(a-1)/2} e^{(x+y)/2} \frac{\partial}{\partial x} \delta(x - y), \\ C_{2n}(x) &= \tilde{R}_{2n}(x) - (2n)(a + 2n)\tilde{R}_{2n-2}(x), \\ C_{2n+1}(x) &= \tilde{R}_{2n+1}, \\ \tilde{r}_n &= (2n + 1)!(a + 2n + 1)!. \end{aligned} \tag{4.32}$$

An interpolation between Case I and Case II was studied in [24].

(3) *Jacobi* [5, 6, 8, 22, 23] The measure is

$$\int d\mu_1(x) = \int_{-1}^1 dx (1 - x)^a (1 + x)^b. \tag{4.33}$$

The corresponding monic orthogonal polynomials are

$$\begin{aligned} C_n(x) &= 2^n n! \frac{(a + b + n)!}{(a + b + 2n)!} P_n^{(a,b)}(x), \\ h_n &= 2^{a+b+2n+1} n! \frac{(a + n)!(b + n)!(a + b + n)!}{(a + b + 2n)!(a + b + 2n + 1)!}, \end{aligned} \tag{4.34}$$

where  $P_n^{(a,b)}(x)$  are the Jacobi polynomials

$$P_n^{(a,b)}(x) = \frac{1}{(1 - x)^a (1 + x)^b} \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left\{ (1 - x)^{a+n} (1 + x)^{b+n} \right\}. \tag{4.35}$$

Case I:

$$\begin{aligned}
 F(x, y) &= (1 - x)^{-(a+1)/2}(1 - y)^{-(a+1)/2}(1 + x)^{-(b+1)/2}(1 + y)^{-(b+1)/2}\text{sgn}(y - x), \\
 \tilde{R}_{2n}(x) &= C_{2n}(x), \\
 \tilde{R}_{2n+1}(x) &= C_{2n+1}(x) - \frac{8n(a + 2n)(b + 2n)(a + b + 2n)}{(a + b + 4n - 1)(a + b + 4n)(a + b + 4n + 1)(a + b + 4n + 2)} \\
 &\quad \times C_{2n-1}(x), \\
 \tilde{r}_n &= 2^{a+b+4n+3}(2n)! \frac{(a + 2n)!(b + 2n)!(a + b + 2n)!}{(a + b + 4n)!(a + b + 4n + 2)!}.
 \end{aligned}
 \tag{4.36}$$

Case II:

$$\begin{aligned}
 F(x, y) &= 2(1 - x)^{-(a-1)/2}(1 - y)^{-(a-1)/2}(1 + x)^{-(b-1)/2}(1 + y)^{-(b-1)/2} \frac{\partial}{\partial x} \delta(x - y), \\
 C_{2n}(x) &= \tilde{R}_{2n}(x) - \frac{8n(a + 2n)(b + 2n)(a + b + 2n)}{(a + b + 4n + 1)(a + b + 4n)(a + b + 4n - 1)(a + b + 4n - 2)} \tilde{R}_{2n-2}(x), \\
 C_{2n+1}(x) &= \tilde{R}_{2n+1}(x), \\
 \tilde{r}_n &= 2^{a+b+4n+3}(2n + 1)! \frac{(a + 2n + 1)!(b + 2n + 1)!(a + b + 2n + 1)!}{(a + b + 4n + 1)!(a + b + 4n + 3)!}.
 \end{aligned}
 \tag{4.37}$$

An interpolation between Case I and Case II was studied in [24].

(4) *Symmetric Hahn* [14] The measure is

$$\int d\mu_1(x) = \sum_{x=-\infty}^{\infty} \frac{1}{[(\frac{L}{2} + x)! (\frac{L}{2} - x)!]^2},
 \tag{4.38}$$

where  $x$  is an integer when  $L$  is an even integer, while it is a half odd integer when  $L$  is odd. The corresponding monic orthogonal polynomials are

$$\begin{aligned}
 C_n(x) &= (-1)^n \frac{(L!)^2(2L - 2n + 1)!}{\{(L - n)!\}^2(2L - n + 1)!} Q_n^{(-L-1, -L-1)}\left(x + \frac{L}{2}; L\right), \\
 h_n &= \frac{n!(2L - 2n + 1)!(2L - 2n)!}{(2L - n + 1)!\{(L - n)!\}^4},
 \end{aligned}
 \tag{4.39}$$

where  $Q_n^{(a,b)}(x; L)$  are the Hahn polynomials [25]

$$Q_n^{(a,b)}(x; L) = \sum_{k=0}^L \frac{(-n)_k(n + a + b + 1)_k(-x)_k}{(a + 1)_k(-L)_k} \frac{1}{k!},
 \tag{4.40}$$

where  $(a)_n = (a + n - 1)!/(a - 1)!$ . If  $F(x, y)$  is given by

$$F(x, y) = \left(\frac{L}{2} + x\right)! \left(\frac{L}{2} - x\right)! \left(\frac{L}{2} + y\right)! \left(\frac{L}{2} - y\right)! \text{sgn}(y - x),$$

then

$$\begin{aligned}\tilde{R}_{2n}(x) &= C_{2n}(x), \\ \tilde{R}_{2n+1}(x) &= C_{2n+1}(x) - \frac{n(L-n+1)(L-2n)(L-2n+1)}{(2L-4n+3)(2L-4n+1)} C_{2n-1}(x), \\ \tilde{r}_n &= \frac{(2n)!(2L-4n+1)!(2L-4n)!}{2(2L-2n+1)!(L-2n-1)! \{(L-2n)!\}^3}.\end{aligned}\quad (4.41)$$

(5) *Discrete Chebyshev* The measure is

$$\int d\mu_1(x) = \sum_{x=0}^L, \quad (4.42)$$

where  $x$  is an integer. The corresponding monic orthogonal polynomials are

$$\begin{aligned}C_n(x) &= (-1)^n \frac{L!(n!)^2}{(L-n)!(2n)!} \mathcal{Q}_n^{(0,0)}(x; L), \\ h_n &= \frac{(L+n+1)!(n!)^4}{(L-n)!(2n+1)!(2n)!},\end{aligned}\quad (4.43)$$

where  $\mathcal{Q}_n^{(a,b)}(x; L)$  are the Hahn polynomials (4.40). If  $F(x, y)$  is given by

$$F(x, y) = \operatorname{sgn}(y - x),$$

then

$$\begin{aligned}C_{2n}(x) &= \tilde{R}_{2n}(x) - \frac{n(2n-1)(L-2n+1)(L+2n+1)}{2(4n-1)(4n+1)} \tilde{R}_{2n-2}(x), \\ C_{2n+1}(x) &= \tilde{R}_{2n+1}(x), \\ \tilde{r}_n &= 2 \frac{(L+2n+2)!(2n)! \{(2n+1)!\}^3}{(L-2n-1)!(4n+2)!(4n+3)!}.\end{aligned}\quad (4.44)$$

(6) *Discrete exponential* [19] The measure is

$$\int d\mu_1(x) = \sum_{x=0}^{\infty} q^x, \quad (4.45)$$

where  $x$  is an integer. The corresponding monic orthogonal polynomials are

$$C_n(x) = \frac{q^n n!}{(q-1)^n} M_n(x; 1, q), \quad h_n = \frac{q^n (n!)^2}{(1-q)^{2n+1}}, \quad (4.46)$$

where  $M_n(x; c, q)$  are the Meixner polynomials [25]

$$M_n(x; c, q) = {}_2F_1 \left( \begin{matrix} -n, -x \\ c \end{matrix} \middle| 1 - \frac{1}{q} \right). \quad (4.47)$$

If  $F(x, y)$  is given by

$$F(x, y) = q^{-(x+y)/2} \alpha^{|y-x|/2} \operatorname{sgn}(y - x),$$



then

$$\begin{aligned}
 \tilde{R}_{2n}(x) &= C_{2n}(x) + \sum_{k=0}^{n-1} \frac{(2n)!}{(2k)!} \frac{q^{n-k}}{(1-q)^{2n-2k}} C_{2k}(x) \\
 &\quad - \frac{\sqrt{\alpha} - \sqrt{q}}{1 - \sqrt{\alpha q}} \sum_{k=0}^{n-1} \frac{(2n)!}{(2k+1)!} \frac{q^{n-k-(1/2)}}{(1-q)^{2n-2k-1}} C_{2k+1}(x), \\
 \tilde{R}_{2n+1}(x) &= C_{2n+1}(x) - \frac{\sqrt{\alpha} - \sqrt{q}}{1 - \sqrt{\alpha q}} \frac{\sqrt{q}}{1-q} (2n+1) C_{2n}(x), \\
 \tilde{r}_n &= \frac{(2n)!(2n+1)! \sqrt{\alpha q}^{2n+(1/2)}}{(1-q)^{4n+1} (1 - \sqrt{\alpha q})^2}.
 \end{aligned}
 \tag{4.48}$$

### 5 Relevance to Random Matrix Theory

In this section we explain how the Pfaffian formulas are applied in random matrix theory and other applications. One of the most popular examples is the Gaussian orthogonal ensemble (GOE) [11]. The GOE is an ensemble of  $N \times N$  real symmetric matrices  $X$  with a Gaussian probability distribution function (p.d.f.)

$$P(X)dX \propto \exp\left\{-\frac{1}{2} \text{Tr } X^2\right\} dX,
 \tag{5.1}$$

where

$$dX = \prod_{j=1}^N dX_{jj} \prod_{j < l}^N dX_{jl}.
 \tag{5.2}$$

Let us change the variables from the matrix elements of  $X$  to the eigenvalues  $x_1, x_2, \dots, x_N$  and eigenvector parameters. Integrating out the eigenvector parameters, we find the p.d.f. of the eigenvalues

$$p(x_1, x_2, \dots, x_N) \prod_{j=1}^N dx_j \propto \prod_{j=1}^N e^{-\frac{1}{2}(x_j)^2} \prod_{j < l}^N |x_j - x_l| \prod_{j=1}^N dx_j.
 \tag{5.3}$$

Therefore, we obtain

$$p(x_1, x_2, \dots, x_N) \propto \prod_{j=1}^N e^{-\frac{1}{2}(x_j)^2} \prod_{j > l}^N (x_j - x_l) \text{Pf}[\text{sgn}(x_l - x_j)]_{j,l=1,2,\dots,N}
 \tag{5.4}$$

for even  $N$  and

$$p(x_1, x_2, \dots, x_N) \propto \prod_{j=1}^N e^{-\frac{1}{2}(x_j)^2} \prod_{j > l}^N (x_j - x_l) \text{Pf} \begin{bmatrix} [\text{sgn}(x_l - x_j)]_{j,l=1,2,\dots,N} & [1]_{j=1,2,\dots,N} \\ -[1]_{l=1,2,\dots,N} & 0 \end{bmatrix}
 \tag{5.5}$$

for odd  $N$ . These are clearly special cases of the general formulas (1.4) and (1.5) with  $M = 1$ . The measure and  $F(x, y)$  are given in Case I of (1) *Hermite* in previous section.

Therefore the correlation functions of the eigenvalues

$$\rho(x_1, \dots, x_k) = \frac{N!}{(N - k)!} \int \prod_{j=k+1}^N dx_j p(x_1, x_2, \dots, x_N) \tag{5.6}$$

can be evaluated in Pfaffian forms.

The ensemble of self-dual real quaternion random matrices with a Gaussian p.d.f. is called the Gaussian symplectic ensemble(GSE). The eigenvalue correlations of the GSE can similarly be analyzed using the formulas given in Case II of (1) *Hermite* in previous section. It is also possible to express the correlation functions for a similar ensemble of antisymmetric hermitian matrices in Pfaffian forms: the derivation of the corresponding skew orthogonal polynomials is illustrated in [21].

The eigenvalue distributions of the matrices of the form  $A^\dagger A$ , where  $A$  are  $M \times N$  rectangular random matrices with Gaussian-distributed elements, are often important in applications. Here  $A^\dagger$  is the hermitian conjugate of  $A$ . When  $A$  are real matrices, we can employ the formulas in Case I of (2) *Laguerre* to write the correlation functions in Pfaffian forms. When  $A$  are real quaternion matrices, Pfaffian expression for the correlation functions are deduced from the formulas in Case II of (2) *Laguerre*.

Hereafter we assume that  $N$  is even for simplicity. The Pfaffian formulas are also useful in analyzing the dynamical correlation among the eigenvalues of the matrix Brownian motion model. The matrix Brownian motion model was also introduced by Dyson [26]. He considered an  $N \times N$  hermitian random matrix  $X$  with the p.d.f.

$$P(X^{(0)}; X; \tau)dX \propto \exp\left[-\frac{\text{Tr}\{(X - e^{-\tau} X^{(0)})^2\}}{1 - e^{-2\tau}}\right]dX \tag{5.7}$$

depending on the parameter  $\tau$ . Here  $X^{(0)}$  is another matrix whose symmetry belongs to a subclass of the symmetry of  $X$ . Note that  $X$  is equated with  $X^{(0)}$  at  $\tau = 0$ . The measure is chosen to be

$$dX = \prod_{j=1}^N dX_{jj} \prod_{j<l}^N d\text{Re } X_{jl} d\text{Im } X_{jl}. \tag{5.8}$$

Dyson showed that the eigenvalue p.d.f.  $p$  of  $X$  satisfies the Fokker–Planck equation

$$\frac{\partial p}{\partial \tau} = \mathcal{L}p, \quad \mathcal{L} = \frac{1}{2} \sum_{j=1}^N \frac{\partial}{\partial x_j} e^{-2w} \frac{\partial}{\partial x_j} e^{2w}, \tag{5.9}$$

where  $x_1, x_2, \dots, x_N$  are the eigenvalues of  $X$  and

$$w = \frac{1}{2} \sum_{j=1}^N x_j^2 - \sum_{j<l}^N \log |x_j - x_l|. \tag{5.10}$$

In order to solve the Fokker–Planck equation, we need to specify the initial condition. For example, let us suppose that  $X^{(0)}$  is a GOE random matrix. Then the Fokker–Planck equation can exactly be solved and the p.d.f.  $p$  of the eigenvalues  $x_1^{(m)}, x_2^{(m)}, \dots, x_N^{(m)}$  at  $\tau = \tau_m$ ,  $m = 1, 2, \dots, M$ , can be explicitly derived. Using the notation  $\mathbf{x}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \dots, x_N^{(m)})$ ,

we find

$$\begin{aligned}
 p(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)}) & \prod_{m=1}^M \prod_{j=1}^N dx_j^{(m)} \\
 & \propto \prod_{j=1}^N e^{-(x_j^{(M)})^2/2} \prod_{j>l}^N (x_j^{(M)} - x_l^{(M)}) \text{Pf}[F(x_j^{(1)}, x_l^{(1)}; \tau_1)]_{j,l=1,2,\dots,N} \\
 & \times \prod_{m=1}^{M-1} \det[g(x_j^{(m+1)}, x_l^{(m)}; \tau_{m+1} - \tau_m)]_{j,l=1,2,\dots,N} \prod_{m=1}^M \prod_{j=1}^N dx_j^{(m)}. \tag{5.11}
 \end{aligned}$$

Here

$$F(x, y; \tau) = \int_{-\infty}^{\infty} dz' \int_{-\infty}^{z'} dz \{g(x, z; \tau)g(y, z'; \tau') - g(y, z; \tau')g(x, z'; \tau)\} \tag{5.12}$$

with

$$g(x, y; \tau) = e^{-(x^2+y^2)/2} \sum_{j=0}^{\infty} \frac{C_j(x)C_j(y)}{h_j} e^{-\gamma_j \tau}, \tag{5.13}$$

where  $C_n(x)$  and  $h_n$  are defined in (4.24) and  $\gamma_j = j + (1/2)$ . The dynamical correlation functions

$$\begin{aligned}
 & \rho(x_1^{(1)}, \dots, x_{k_1}^{(1)}; x_1^{(2)}, \dots, x_{k_2}^{(2)}; \dots; x_1^{(M)}, \dots, x_{k_M}^{(M)}) \\
 & = \frac{(N!)^M}{\prod_{m=1}^M (N - k_m)!} \int \prod_{j=k_1+1}^N dx_j^{(1)} \int \prod_{j=k_2+1}^N dx_j^{(2)} \dots \\
 & \times \int \prod_{j=k_M+1}^N dx_j^{(M)} p(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)}) \tag{5.14}
 \end{aligned}$$

are then calculated in Pfaffian forms. This Pfaffian formula is known to be useful in analyzing the asymptotic behavior of the dynamical correlation functions [11, 27, 28].

The Pfaffian formulas can also be applied to multi-matrix models in which one  $N \times N$  real symmetric matrix  $X^{(1)}$  is combined to  $N \times N$  hermitian matrices  $X^{(2)}, X^{(3)}, \dots, X^{(M)}$  [13]. The p.d.f. is given by

$$\begin{aligned}
 & P(X^{(1)}, X^{(2)}, \dots, X^{(M)})(dX) \\
 & \propto \exp \left[ -\text{Tr} \left\{ \frac{1}{2} V_1(X^{(1)}) + \sum_{m=2}^{M-1} V_m(X^{(m)}) + \frac{1}{2} V_M(X^{(M)}) \right\} \right] \\
 & \times \exp[\text{Tr}\{c_1 X^{(1)} X^{(2)} + c_2 X^{(2)} X^{(3)} + \dots + c_{M-1} X^{(M-1)} X^{(M)}\}](dX) \tag{5.15}
 \end{aligned}$$

with the measure

$$(dX) = \prod_{j=1}^N dX_{jj}^{(1)} \prod_{j<l}^N dX_{jl}^{(1)} \prod_{m=2}^M \left\{ \prod_{j=1}^N dX_{jj}^{(m)} \prod_{j<l}^N d\text{Re} X_{jl}^{(m)} d\text{Im} X_{jl}^{(m)} \right\}. \tag{5.16}$$

As before we like to change the variables and integrate out the eigenvector parameters. For that purpose, Harish–Chandra’s integral formula can be utilized: for  $N \times N$  diagonal matrices  $A$  and  $B$  with diagonal elements  $a_1, a_2, \dots, a_N$  and  $b_1, b_2, \dots, b_N$ , respectively, the integral over the group of  $N \times N$  unitary matrices  $U$  can be evaluated as [29]

$$\int \exp\left\{-\frac{1}{t} \text{Tr}(A - UBU^\dagger)^2\right\} dU \propto \frac{1}{\prod_{j < l}^N (a_j - a_l)(b_j - b_l)} \det\left[\exp\left\{-\frac{1}{t}(a_j - b_l)^2\right\}\right]_{j,l=1,2,\dots,N}. \tag{5.17}$$

Let us write the eigenvalues of  $X^{(m)}$  as  $x_1^{(m)}, x_2^{(m)}, \dots, x_N^{(m)}$  and define the vectors  $\mathbf{x}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \dots, x_N^{(m)})$ . Then, after integrating out the eigenvector parameters, we obtain the p.d.f. of the eigenvalues

$$p(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)}) \prod_{m=1}^M \prod_{j=1}^N dx_j^{(m)} \propto \prod_{j>l}^N (x_j^{(M)} - x_l^{(M)}) \text{Pf}[\text{sgn}(x_l^{(1)} - x_j^{(1)})]_{j,l=1,2,\dots,N} \times \prod_{m=1}^{M-1} \det[g^{(m)}(x_j^{(m+1)}, x_l^{(m)})]_{j,l=1,2,\dots,N} \prod_{m=1}^M \prod_{j=1}^N dx_j^{(m)} \tag{5.18}$$

with

$$g^{(m)}(x, y) = \exp\left\{-\frac{1}{2}V_{m+1}(x) - \frac{1}{2}V_m(y) + c_mxy\right\}. \tag{5.19}$$

The correlation functions for the multi-matrix models are defined by the formula (5.14) with the p.d.f.  $p$  replaced by (5.18). Comparison of the above formula with (1.4) reveals that the correlation functions are written in Pfaffian forms.

Recently, through the discretization of random matrix ensembles, it turned out that the dynamical correlation functions for stochastic motions of interacting many particles were sometimes written in Pfaffian forms. One of the simplest examples is given by the vicious walkers [14, 15] on the lattice  $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ . Let us consider  $N$  simple symmetric random walkers who start from the positions  $2X_1 < 2X_2 < \dots < 2X_N$ ,  $X_j \in \mathbf{Z}$ , at time  $k = 0$ . The positions  $S_j^{(k)}$  of the  $j$ th walkers at time  $k \geq 0$  are restricted by the nonintersecting condition

$$S_1^{(k)} < S_2^{(k)} < \dots < S_N^{(k)}, \quad 1 \leq k \leq K. \tag{5.20}$$

Let  $V(X_1, \dots, X_N; Y_1, \dots, Y_N)$  be the probability that the  $N$  walkers come to the positions  $2Y_1 < 2Y_2 < \dots < 2Y_N$ ,  $Y_j \in \mathbf{Z}$ , at an even integer time  $K$ . Then the nonintersecting condition implies a determinant formula

$$V(X_1, \dots, X_N; Y_1, \dots, Y_N) = \frac{1}{2^{KN}} \det\left[\left(K/2 + X_l - Y_j\right)\right]_{j,l=1,2,\dots,N}. \tag{5.21}$$

Hereafter we employ the scaled variables  $t = K/L$ ,  $x_j = 2X_j/\sqrt{L}$  and  $y_j = 2Y_j\sqrt{L}$ , with which a simplification occurs in the scaling limit  $L \rightarrow \infty$ .

We consider the nonintersecting motion in the scaled time interval  $0 < t \leq T$  and suppose that the scaled positions of all the walkers are initially at the origin. Let us denote the scaled positions of the  $j$ th walker at a scaled time  $t_m$  by  $x_j^{(m)}$  and define  $\mathbf{x}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \dots, x_N^{(m)})$ . Then, from the determinant formula (5.21), the p.d.f. of the walkers is evaluated in the limit  $L \rightarrow \infty$  as

$$\begin{aligned}
 & p(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(M)}) \prod_{m=1}^M \prod_{j=1}^N dx_j^{(m)} \\
 & \propto \int \prod_{j=1}^N dx_j^{(M+1)} \prod_{j>l}^N (x_j^{(1)} - x_l^{(1)}) \text{Pf}[\text{sgn}(x_l^{(M+1)} - x_j^{(M+1)})]_{j,l=1,2,\dots,N} \\
 & \times \prod_{m=1}^M \det[g^{(m)}(x_j^{(m)}, x_l^{(m+1)})]_{j,l=1,\dots,N} \prod_{m=1}^M \prod_{j=1}^N dx_j^{(m)}, \tag{5.22}
 \end{aligned}$$

where

$$\begin{aligned}
 g^{(1)}(x, y) &= \frac{e^{-x^2/(2t_1)} e^{-(x-y)^2/(2(t_2-t_1))}}{\sqrt{2\pi t_1} \sqrt{2\pi(t_2-t_1)}}, \\
 g^{(m)}(x, y) &= \frac{e^{-(x-y)^2/(2(t_{m+1}-t_m))}}{\sqrt{2\pi(t_{m+1}-t_m)}}, \quad 2 \leq m \leq M
 \end{aligned} \tag{5.23}$$

with  $t_{M+1} = T$ . The dynamical correlation functions can be defined by the formula (5.14) with the p.d.f.  $p$  replaced by (5.22). Then we can apply the Pfaffian formulas in Sect. 2. The asymptotic behavior of the correlation functions can be analyzed by using the Pfaffian formulas [15].

In this article we focused on the applications of the Pfaffian to hermitian random matrices and related stochastic models. It should be remarked that the Pfaffian is also useful in evaluating the eigenvalue correlations of unitary [30], non-self-dual real quaternion [11] and asymmetric real [31] random matrices.

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